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METHOD OF CALCULATING STEADY-STATE FLOWS OF A VISCOUS FLUID WITH FREE BOUNDARY IN VORTEX–STREAM FUNCTION VARIABLES

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Computations of the motion of a viscous incompressible fluid with a free boundary in vortex–stream function variables entail difficulties in the implementation of boundary conditions on the free surface. A new approach to the formulation of boundary conditions, which takes into account the specifics in giving these conditions on the free boundary is proposed. An efficient numerical method of calculating steady-state flows of a fluid is developed and implemented. Model computations for problems that have exact solutions are performed.

The main difficulties in the solution of problems in domains having a free boundary are associated with the fact that the boundary conditions are given on the previously unknown surfaces to be found in the process of solution of the problem. In the present paper, the focus is on two basic aspects:

(a) development of a mathematical model of motion of a viscous fluid with a free boundary, which includes the governing equations describing this motion, and the boundary conditions;

(b) implementation of the proposed method in a numerical solution of the problem.

1. Mathematical Model. Let a viscous incompressible fluid of density ρ, kinematic viscosity ν, and with coefficient of surface tension σ occupy the domain GB: $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant f(x)$, where $f(x)$ is the unknown free surface (Fig. 1). The gravity vector $\mathbf{g}$ is parallel to the y axis and is pointing downward. The system of equations which describe the fluid motion in the variables ω (vortex) and ψ (stream function) is of the form

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}\left(\omega \frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\omega \frac{\partial \psi}{\partial x}\right) = \frac{1}{\mathrm{Re}} \Delta\omega; \tag{1.1}$$

$$\Delta\psi = -\omega. \tag{1.2}$$

Here $\mathrm{Re} = v_0 h_0/\nu$ is the Reynolds number; the quantity ρv_0^2 is used as the pressure scale, and v_0 and h_0 are the characteristic scales of velocity and depth of the fluid. The stream function is introduced by the relations

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{1.3}$$

The boundary conditions at the lateral and lower boundaries of the domain GB are assumed to be specified. The kinematic condition and the continuity conditions for the normal and tangential components of the stress vector are set at the free boundary.

The kinematic condition is of the form

$$f_t = v - f_x u. \tag{1.4}$$

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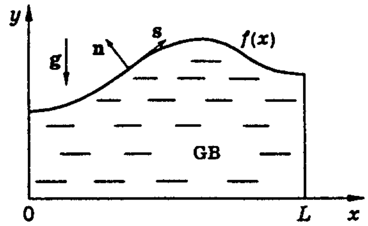

Fig. 1

Defining the vectors of the normal and the tangent to the free surface $f(x)$ at each point of this surface as

$$\mathrm{n} = \left\{ \frac{-f_x}{\sqrt{1+f_x{}^2}}, \frac{1}{\sqrt{1+f_x{}^2}} \right\}, \qquad \mathrm{s} = \left\{ \frac{1}{\sqrt{1+f_x{}^2}}, \frac{f_x}{\sqrt{1+f_x{}^2}} \right\},$$

we write the continuity condition for the normal component of the stress vector on the free surface:

$$P - P_0 = -\frac{\mathrm{Ca}^{-1}}{\mathrm{Re}} \frac{1}{R} + \frac{2}{\mathrm{Re}(1+f_x^2)} \left[f_x^2 \frac{\partial u}{\partial x} - f_x \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial y} \right], \quad \frac{1}{R} = \frac{f_{xx}}{\sqrt{(1+f_x^2)^3}}. \tag{1.5}$$

Here P is the fluid pressure on the free surface, P_0 is the external pressure (e.g., atmospheric pressure), R is the curvature radius of $f(x)$, and $\mathrm{Ca} = \rho v_0 \nu / \sigma$ is the parameter that is called a capillary number.

The continuity equation for the tangential component of the stress vector is of the form

$$2f_x \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + (1 - f_x^2) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0. \tag{1.6}$$

Excluding the pressure in the Navier–Stokes equations, which are written in variables u, v, and P by means of cross differentiation, we obtain, generally speaking, a fourth-order equation for the stream function ψ. After that, Eq. (1.1) can be regarded as the writing of this equation, where ω is the function defined by Eq. (1.2). In this case, Eq. (1.1) expresses the law of displacement of the vortex ω. With the pressure omitted, a similar procedure can be performed for boundary conditions as well [1]. Using (1.3), we obtain, for the stream function at the domain's boundary, a rather cumbersome third-order equation containing mixed derivatives [2, 3]. It is difficult to solve and analyze such an equation, not to mention the vast amount of preliminary work that should be done by a researcher, especially if the determination of the free boundary is only a fragment of a large problem. It makes sense to introduce an intermediate function which obeys, as ω in the governing equations, the law obtained on the basis of the initial equations.

We introduce the notation. Let

$$v_s = \left. \frac{\partial \psi}{\partial n} \right|_{y=f(x)} \tag{1.7}$$

be the fluid velocity in the direction of the tangent on the free surface. Since only the stationary fluid motion is considered and the free surface is a stream line in this case, the relation [4]

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s} = -\frac{\partial P}{\partial s} - \frac{1}{\mathrm{Re}} \frac{\partial \omega}{\partial n} - \frac{\mathrm{G} f_x}{\sqrt{1+f_x^2}} \tag{1.8}$$

is satisfied on the free surface. Here and below, the time t is regarded as a parameter, $\mathrm{G} = h_0 g / v_0^2$ is the Galilei number, and g is the acceleration of gravity.

As in [1], Eq. (1.5) can be presented as follows:

$$P - P_0 = -\frac{\mathrm{Ca}^{-1}}{\mathrm{Re}} \frac{1}{R} - \frac{2}{\mathrm{Re}} \frac{\partial v_s}{\partial s}. \tag{1.9}$$

Differentiating (1.9) with respect to s and substituting the result into (1.8), we obtain

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s} = \frac{2}{\mathrm{Re}} \frac{\partial^2 v_s}{\partial s^2} + D, \tag{1.10}$$

where

$$D = \frac{\partial}{\partial s}\Big(- P_0 + \frac{\mathrm{Ca}^{-1}}{\mathrm{Re}} \frac{1}{R}\Big) - \frac{1}{\mathrm{Re}} \frac{\partial \omega}{\partial n} - \frac{\mathrm{G} f_x}{\sqrt{1 + f_x^2}}.$$

With the flow stationarity taken into account, one can derive an explicit relation for a vortex on the free surface [1] from condition (1.6):

$$\omega = \frac{2}{R} v_s.$$

The boundary condition for the stream function follows from (1.7), where v_s is the solution of Eq. (1.10), which possesses remarkable properties such as the divergent form relative to v_s and the simplicity of the solution. In addition, Eqs. (1.7) and (1.10) can be considered as an analog of the two-field method written for a free surface. To find the free surface, we use Eq. (1.4), which may be written in the form

$$f_t + \sqrt{1 + f_x^2}\, \frac{\partial \psi}{\partial s} = 0.$$

2. Method of Solution. We shall map the domain GB onto the rectangle $0 \leqslant \xi \leqslant L$, $0 \leqslant \eta \leqslant 1$ by means of the following transformation:

$$x = \xi, \qquad y = f(\xi)\eta.$$

After that, all the boundaries of the domain GB, including the free surface, coincide with the coordinate lines of the new grid, and each equation in (1.1) and (1.2) can be represented in the form

$$\frac{\partial \Phi}{\partial t} = \frac{1}{Bf}\Big[\frac{\partial}{\partial \xi}\Big(B_{11} \frac{\partial \Phi}{\partial \xi} + B_{12} \frac{\partial \Phi}{\partial \eta} - A\Phi \frac{\partial \psi}{\partial \eta}\Big) + \frac{\partial}{\partial \eta}\Big(B_{12} \frac{\partial \Phi}{\partial \xi} + B_{22} \frac{\partial \Phi}{\partial \eta} + A\Phi \frac{\partial \psi}{\partial \xi}\Big)\Big] + F. \tag{2.1}$$

Here

$$B_{11} = f(\xi), \qquad B_{12} = -f_\xi \eta, \qquad B_{22} = (1 + B_{12}^2)/f(\xi). \tag{2.2}$$

We note that $B_{11}B_{22} - B_{12}^2 = 1$.

If $\Phi = \omega$, then $B = \mathrm{Re}$, $A = \mathrm{Re}$, and $F = 0$. If $\Phi = \psi$, then $B = 1/\lambda$, $A = 0$, and $F = \lambda\omega$ (λ is the iteration parameter introduced in the solution of the Poisson equation for ψ).

We introduce the following notation:

$$U(\Phi) = B_{11} \frac{\partial \Phi}{\partial \xi} + B_{12} \frac{\partial \Phi}{\partial \eta} - A\Phi \frac{\partial \psi}{\partial \eta}, \quad V(\Phi) = B_{12} \frac{\partial \Phi}{\partial \xi} + B_{22} \frac{\partial \Phi}{\partial \eta} + A\Phi \frac{\partial \psi}{\partial \xi}.$$

Equation (2.1) takes the form

$$\frac{\partial \Phi}{\partial t} = \frac{1}{Bf}[U_\xi(\Phi) + V_\eta(\Phi)] + F, \tag{2.3}$$

and the boundary conditions on the free surface ($\eta = 1$) can be written as follows:

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial \xi} = \frac{2}{\mathrm{Re}} \frac{\partial}{\partial \xi}\Big(\frac{1}{\sqrt{1 + f_\xi^2}} \frac{\partial v_s}{\partial \xi}\Big) + D, \tag{2.4}$$

where

$$D = \frac{\partial}{\partial \xi}\Big(- P_0 + \frac{\mathrm{Ca}^{-1}}{\mathrm{Re}} \frac{1}{R}\Big) - \frac{1}{\mathrm{Re}}\Big(B_{12} \frac{\partial \omega}{\partial \xi} + B_{22} \frac{\partial \omega}{\partial \eta}\Big) - \mathrm{G} f_\xi;$$

$$\frac{\partial \psi}{\partial \eta} = \frac{v_s\sqrt{1 + f_\xi^2}}{B_{22}}; \tag{2.5}$$

$$\omega = \frac{2}{R} v_s; \tag{2.6}$$

$$\frac{\partial f}{\partial t} + \frac{\partial \psi}{\partial \xi} = 0. \tag{2.7}$$

It is worth noting that the kinematic condition (2.7) now expresses the law of mass conservation.

We shall search for the solution of Eq. (2.3) at each time step using the scheme of a stabilizing correction [5] taken in the form

$$\begin{aligned} \frac{\Phi^{k+1/2} - \Phi^k}{\tau} &= \frac{1}{Bf}\left[U_\xi^k(\Phi) + V_\eta^{k+1/2}(\Phi)\right] + F, \\ \frac{\Phi^{k+1} - \Phi^{k+1/2}}{\tau} &= \frac{1}{Bf}\left[U_\xi^{k+1}(\Phi) - U_\xi^k(\Phi)\right]. \end{aligned} \tag{2.8}$$

Here

$$\begin{aligned} V^{k+1/2}(\Phi) &= B_{12}\Phi_\xi^k + B_{22}\Phi_\eta^{k+1/2} + A\Phi^{k+1/2}\frac{\partial \psi}{\partial \xi}, \\ U^{k+1}(\Phi) &= B_{11}\Phi_\xi^{k+1} + B_{12}\Phi_\eta^{k+1/2} - A\Phi^{k+1}\frac{\partial \psi}{\partial \eta}. \end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) that Eq. (2.3) is first solved in the direction of η and then in the direction of ξ. We shall show that the succession of the directions of the solution (2.3) is different for the functions ψ and ω. It is important to note that the relations for mixed derivatives are taken from the previous semi-step. The stabilizing-correction scheme is classified as an economic difference scheme with fractional steps, in which the first step produces a complete approximation of the equation and the next step is a correction, whose purpose is to improve the stability.

To implement the scheme (2.8), (2.9), a rectangular computational grid is constructed in the standard manner in a rectangle which corresponds to the transformed domain GB:

$$\begin{aligned} &\xi_n = (n-1)\Delta\xi, \quad \Delta\xi = L/NB, \qquad n = 1, \ldots, NN, \quad NN = NB + 1, \\ &\eta_m = (m-1)\Delta\eta, \quad \Delta\eta = 1/MB, \qquad m = 1, \ldots, MM, \quad MM = MB + 1. \end{aligned}$$

We approximate differential expressions of the type $(a_{11}\Phi_\xi)_\xi$, $(a_{22}\Phi_\eta)_\eta$, $(a_1\Phi)_\xi$, and $(a_2\Phi)_\eta$ with second-order accuracy by the finite-difference analogs Λ_{11}, Λ_{22}, Λ_1, and Λ_2, which have the traditional representation [5, 6]. To approximate the mixed derivative, for example, $(a_{12}\Phi_\xi)_\eta$, according to [7], we use the operator

$$\Lambda_{12}\Phi = \frac{(a_{12})_{n,m+1}(\Phi_{n+1,m+1} - \Phi_{n-1,m+1}) - (a_{12})_{n,m-1}(\Phi_{n+1,m-1} - \Phi_{n-1,m-1})}{4\Delta\xi\Delta\eta}.$$

The operator $\Lambda_{21}\Phi$ is determined similarly. The scheme (2.8) and (2.9) approximates (2.3) with accuracy $O(\tau + h^2)$.

After the derivatives are replaced by the corresponding finite differences in (2.8) and their difference analogs from (2.9) are substituted for $V_\eta(\Phi)$ and $U_\xi(\Phi)$, at each time semi-step, for all the internal points ($n = 2, \ldots, NB$ and $m = 2, \ldots, MB$) we obtain a system of linear difference equations relative to $\Phi(\xi_n, \eta_m)$. The system has a three-diagonal structure with dominance of the diagonal matrix elements, and it can be effectively solved by the sweep method, with allowance for the boundary conditions.

Equation (2.4) belongs to equations of the Burgers–Hopf type with the right-hand side. As was shown in [7, 8], in approximating and deriving a numerical solution, the representation of such an equation in a conservative form is of significance. We write (2.4) in the form

$$\frac{\partial v_s}{\partial t} + \frac{1}{2}\frac{\partial}{\partial \xi}(v_s)^2 = \frac{2}{\mathrm{Re}}\frac{\partial}{\partial \xi}\left(\frac{1}{\sqrt{1+f_\xi^2}}\frac{\partial v_s}{\partial \xi}\right) + D$$

and approximate the second term on the left-hand side as follows [8]:

$$\frac{1}{2}\frac{\partial}{\partial\xi}(v_s)^2 \to \frac{1}{2}\frac{(v_s^2)_{n+1}^{k+1}-(v_s^2)_{n-1}^{k+1}}{2\Delta\xi} \to \frac{1}{2}\frac{(v_s)_{n+1}^{k}\,(v_s)_{n+1}^{k+1}-(v_s)_{n-1}^{k}(v_s)_{n-1}^{k+1}}{2\Delta\xi}.$$

The first term on the right-hand side is approximated in the upper time layer $(k+1)$ by the difference operator Λ_{11}. The remaining differential operators also have the traditional representation. For a derivative with respect to η, a one-sided second-order approximation is used at the boundary of the domain, the value of $\partial\omega/\partial n$ being taken from the previous time step. The system of difference equations obtained has a three-diagonal structure, and it is solved by the sweep method. The boundary conditions for the solution (2.4) can be determined from the boundary conditions for ψ specified at the lateral boundaries of the domain GB and also from the physical formulation of the problem.

3. Calculation of the Vortex and the Stream Function. The central idea of the proposed method of solving Eq. (2.3) with boundary conditions (2.4)–(2.7) can be expressed as follows. We separate, among the basic grid functions $\omega(\xi_n,\eta_m)$ and $\psi(\xi_n,\eta_m)$, a group of unknowns, which are called parameters and via which one can express the main unknown at all the internal points of the domain considered. For example, this will be the values of ω at the domain's boundaries for the vortex and the values of ψ at the near-the-boundary points for the stream function. Having derived relations containing only these parameters, we solve these relations using the boundary conditions. With the parameters found, we restore all the values of the major unknowns at all the points of the domain.

We shall describe the stages of a transition from the kth to the $(k+1)$th time layer. Let the stream-function and vortex fields be known at time $t_k = k\tau$. The position of the free surface at which these fields were found is assumed to be known as well.

Stage I. We shall find the new position of the free surface which corresponds to the moment $t_{k+1} = (k+1)\tau$ from Eq. (2.7) and the matrix of the coefficients B_{11}, B_{12}, and B_{22} from formulas (2.2). Solving Eq. (2.4), we find the boundary conditions for ψ and ω on the free surface according to formulas (2.5) and (2.6), respectively.

Stage II. Equation (2.3) is first solved for the vortex in the direction of ξ. The boundary condition for the vortex is taken from the lower time layer. The derived systems of difference equations are solved by the sweep method. After that, Eq. (2.3) is solved in the direction of η. The solution of the systems of difference equations is searched for by the parameter-matching method [9], i.e.,

$$\omega(n,m) = P(n,m)\omega^*(n) + Q(n,m)\omega^{**}(n) + \bar{\omega}(n,m), \tag{3.1}$$

where $\omega^*(n)$ are the still unknown vortex values at the lower boundary of the domain $(\eta = 0)$, $\omega^{**}(n)$ is the vortex value on the free surface $(\eta = 1)$, and $P(n,m)$, $Q(n,m)$, and $\bar{\omega}(n,m)$ are the known two-dimensional massives; note that $P(n,1) = 1$, $Q(n,1) = 0$, $\bar{\omega}(n,1) = 0$, $P(n,MM) = 0$, $Q(n,MM) = 1$, and $\bar{\omega}(n,MM) = 0$. We keep in mind the massives $P(n,m)$, $Q(n,m)$, and $\bar{\omega}(n,m)$ and pass to the next stage.

Stage III. We shall consider a concrete condition at the lower boundary of the domain GB $(\eta = 0)$. Let, for example, the lower boundary be a rigid impenetrable wall in which the boundary conditions are given in the form of the viscous nonslip conditions. Using the Thom formula [7] to relate the vortex and the stream function at the boundary, we can write

$$\psi\Big|_{\eta=0} = 0, \qquad \omega^*(n) = -\frac{2}{(\Delta\eta)^2}\frac{1}{f^2}\psi_{n,2}. \tag{3.2}$$

Relations (2.5) and (2.6) are satisfied at the upper boundary $(\eta = 1)$:

$$\frac{\partial\psi}{\partial\eta}\Big|_{\eta=1} = \frac{v_s\sqrt{1+f_\xi^2}}{B_{22}}\Big|_{\eta=1}, \qquad \omega^{**}(n) = \frac{2}{R}v_s. \tag{3.3}$$

Solving (2.3) for the stream function in the direction of η with allowance for (3.1) and (3.2), for each n we obtain a system of difference equations of the form

$$-a_m\psi_{m-1} + b_m\psi_m - c_m\psi_{m+1} = F_m - A_m\psi_2 \tag{3.4}$$

(for simplicity, n-containing terms are omitted). It is natural to find a solution of (3.4), with allowance for the boundary condition for ψ (3.3), in the form [10]

$$\psi_m = S(m)\psi_2 + T(m)\psi_{MB} + \bar{\psi}(m), \tag{3.5}$$

where ψ_2 and ψ_{MB} are the still unknown functions, and $S(m)$, $T(m)$, and $\bar{\psi}_m$ are the known massives; we note that

$$S(1) = T(1) = \bar{\psi}(1) = 0,$$

$$S(MM) = 0, \quad T(MM) = 1, \quad \bar{\psi}(MM) = \left.\frac{\Delta\eta v_s\sqrt{1+f_\xi^2}}{B_{22}}\right|_{\eta=1}.$$

Substituting successively ψ_2 and ψ_{MB} into the left-hand side of (3.5), we derive a system of two linear equations to define these functions. After that, we restore ψ by formulas (3.5) at all the points of the domain. To simplify the description of the method, the derivative $\partial\psi/\partial\eta$ is approximated over the first order. Clearly, the order of approximation can be increased to any reasonable limits. On the right-hand side of Eq. (3.5) will appear the terms with the ψ_m values that were used for approximation of $\partial\psi/\partial\eta$ at the upper boundary of the domain. Next, Eq. (2.3) is solved in the direction of ξ taking into account the boundary conditions. Stage III is repeated until the condition

$$\max_{n,m}\left|\frac{|\psi_{n,m}^{s+1}| - |\psi_{n,m}^{s}|}{|\psi_{n,m}^{s+1}|}\right| < \varepsilon,$$

where s is the iteration number and ε is the prescribed accuracy, is met.

Stage IV. After the third stage is completed, we find $\omega^*(n)$ by formulas (3.2) and then restore ω by formula (3.1) at all the points of the domain. Thus, the transition to the new time layer is accomplished. After that, the process is repeated until the stationary solution is obtained. This condition is considered to be found if the condition

$$\max_n\left|f_n^{k+K} - f_n^k\right| < \varepsilon$$

(K is the given number of steps and ε is the given accuracy) is satisfied.

Remark. Assuming that the velocity along the normal is $v_n = 0$ and "forcing" the free surface at any moment of time to be the stream function [Eq. (1.8)], we thus introduce a restriction on the class of problems to be solved. It follows that the proposed method allows us to solve only problems that have stable stationary solutions.

4. Examples of Calculation. To check the accuracy and efficiency of a numerical algorithm, it is necessary to use either exact solutions of model equations (if they are available) or approximate solutions that have been well studied and determined by other researchers.

Example 1. We consider the problem proposed in [2], where there are both the exact solution of the linearized system and the approximate solution of the complete equations. Let a viscous incompressible fluid execute the motion in the infinite band $-\infty < x < \infty$, $0 \leqslant y \leqslant f(x)$. The velocity-profile conditions are specified at the lower boundary of the domain $y = 0$, which have the following form in terms of ψ:

$$\psi = \psi_m \sin\frac{2\pi}{L}x, \qquad \frac{\partial\psi}{\partial y} = 0.$$

The upper boundary of the fluid $y = f(x)$ is a free surface. Equations (1.1) and (1.2) are dimensionalized, as was done in [2], by selecting h_0, h_0^2/ν, and ν as the scales of length, time, and stream function, respectively. We shall find the stationary solution of a problem that is subject to the following periodicity condition:

$$\psi(x+L, y) = \psi(x, y), \qquad \omega(x+L, y) = \omega(x, y), \qquad f(x+L, y) = f(x, y).$$

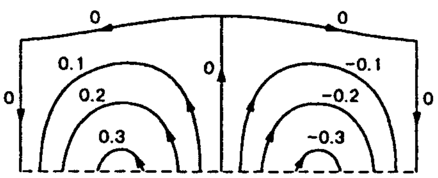

Fig. 2

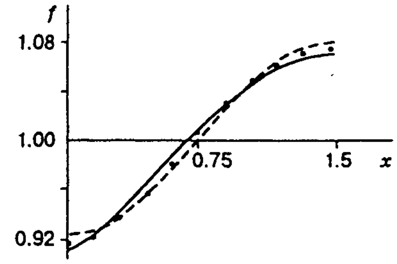

Fig. 3

For these boundary conditions, the linearized problem has the analytical solution

$$f_0 = 1 - \psi_m \frac{2k^2(\cosh k + k\sinh k)}{(\sinh k \cosh k - k)(\mathrm{G} + \mathrm{Ca}^{-1}k^2/\mathrm{Re})} \cos kx, \quad k = \frac{2\pi}{L}. \tag{4.1}$$

The stationary solution turns out to be symmetric owing to the periodicity condition and the symmetry of the velocity profile at the lower boundary. Passing over to the new variables (ξ, η), we can write the boundary conditions for ψ and ω at the left and right boundaries in the form

$$\psi\Big|_{\xi=0} = \omega\Big|_{\xi=0} = 0, \quad \psi\Big|_{\xi=L} = \omega\Big|_{\xi=L} = 0, \quad v_s\Big|_{\xi=0} = v_s\Big|_{\xi=L} = 0.$$

We note that P_0 in (2.4) is assumed to be equal to 0 as well.

The vortex and the stream function are related at the lower boundary ($\eta = 0$) using the Thom approach:

$$\omega_{n,1} = -2\frac{\psi_{n,2} - \psi_{n,1}}{f^2(\Delta\eta)^2} - \left(\frac{\partial^2\psi}{\partial\xi^2}\right)_{n,1}.$$

On the free boundary, the boundary conditions for ψ and ω are specified by relations (2.5) and (2.6), respectively, where v_s is the solution of Eq. (2.4). This method was employed for calculation on various grids. Figure 2 shows isolines of the stream function for a flow computed on a 21×11 grid for $L = 3$, $\mathrm{Ca}^{-1} = 0$, $\psi_m = 0.32$, and $\mathrm{G} = 32$. The curving of the free surface $f(x)$ is depicted in Fig. 3. The dashed curve indicates the graph of the function $f_0(x)$ defined by formula (4.1). The points denote the solution of the problem derived in [2] on a 24×21 grid. The amplitude of curving of the free surface $\delta = \max\limits_x f(x) - \min\limits_x f(x)$ deviates from the quantity δ_0 obtained from (4.1) by less than 1%.

The problem was solved under the assumption that the surface tension of the fluid is zero. It follows from (4.1) that the maximum amplitude of curving of the free surface will be obtained for given G and ψ_m. However, the surface tension plays a very important role in the formation of the free surface, especially in small-sized domains. In addition, surface tension acts as a stabilizer of the solution. This fact was manifested in studying the solution of the problem, which was performed for the same values of L, G, and ψ_m that were considered above, and $0 < \mathrm{Ca}^{-1} \leqslant 100$. The calculation results that were obtained by the author coincide with those obtained in [2].

Example 2. At the initial moment of time, the fluid which occupies the domain GB $\{0 \leqslant x \leqslant L,\ 0 \leqslant y \leqslant 1\}$ is in a state of rest. Here $x = 0$, $x = L$, and $y = 0$ are solid impenetrable walls, and $y = 1$ is the free surface. The source of perturbation is the x-variable domain of external pressure $P_0 = P_0(x)$ applied to the free surface.

The boundary conditions at solid walls are as follows:

$$\psi\Big|_{x=0} = 0,\ \psi\Big|_{x=L} = 0,\ \psi\Big|_{y=0} = 0,\ \frac{\partial\psi}{\partial n}\Big|_{x=0} = \frac{\partial\psi}{\partial n}\Big|_{x=L} = 0,\ \frac{\partial\psi}{\partial n}\Big|_{y=0} = 0,\ v_s\Big|_{x=0} = v_s\Big|_{x=L} = 0.$$

The Thom condition is used to relate the stream function to the vortex at solid walls.

To determine the boundary conditions on the free surface, we solve Eq. (2.4), where P_0 is the given

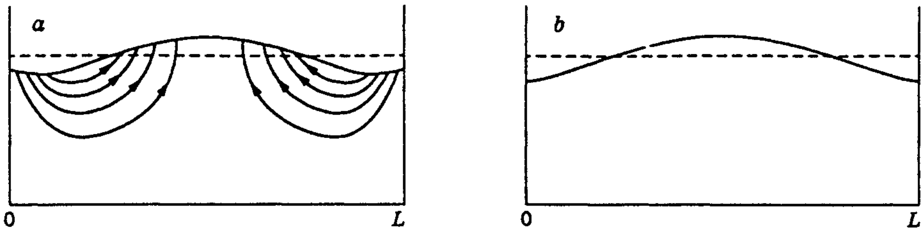

Fig. 4

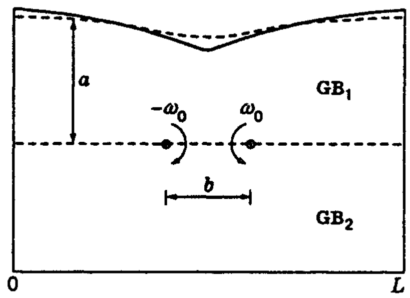

Fig. 5

function. To specify P_0, we use the solution results of the first example. Having solved Problem 1 for $\mathrm{Ca}^{-1} = 0.03$ and $P_0 = 0$, we obtain an explicit pressure distribution on the free surface according to formula (1.9) in the form of the function

$$P\Big|_{\eta=1} = \bar{P}(\xi).$$

Figure 4 shows calculation results for $\mathrm{Ca}^{-1} = 0.03$, G = 32, and $P_0 = 3\bar{P}(\xi)$. The dashed curve indicates the position of the free surface at the initial moment of time, and the solid curves refer to the position of the free boundary and the isolines of the stream function at a certain intermediate moment of time before (Fig. 4a) and at the moment of (Fig. 4b) reaching the position.

Example 3. Let the fluid occupying the domain GB $\{0 \leqslant x \leqslant L,\ 0 \leqslant y \leqslant 2\}$ be in a quiescent state at the initial moment of time. The source of perturbation is rotation of a pair of cylinders which are perpendicular to the plane of a liquid flow whose geometrical dimensions may be ignored. Such a source of perturbation can be treated as giving a pair of vortices inside the domain (Fig. 5):

$$\omega(x_0, 1) = -\omega_0, \qquad \omega(L - x_0, 1) = \omega_0.$$

We specify the boundary conditions at the lower boundary of the domain GB ($y = 0$) in the form of the viscous nonslip conditions

$$\psi\Big|_{y=0} = \frac{\partial\psi}{\partial n}\Big|_{y=0} = 0,$$

and in the form of slip conditions at the left and right boundaries of the domain GB:

$$\psi\Big|_{x=0} = \omega\Big|_{x=0} = 0, \qquad \psi\Big|_{x=L} = \omega\Big|_{x=L} = 0.$$

The simplest method of solving the problem is to divide the domain GB into two subdomains, GB$_1$ and GB$_2$, as was done in [11]. As a result, the source of perturbation is transferred from inside the domain to the boundary which is shared by both subdomains (the dashed curve). The upper subdomain GB$_1$ is mapped onto a rectangle by the formulas described in Sec. 2. A 21×11 calculation grid is constructed in each rectangle, and the problem is solved locally in each subdomain with the following conditions at the common boundary:

$$\omega(n, MM)\Big|_{\mathrm{GB}_2} = \omega(n, 1)\Big|_{\mathrm{GB}_1}, \qquad \frac{\partial\psi}{\partial y}\Big|_{(n,MM),\mathrm{GB}_2} = \frac{\partial\psi}{\partial y}\Big|_{(n,1),\mathrm{GB}_1}.$$

To find the distribution of ω at the common boundary, we solve the equation for the vortex in the direction of ξ in the domain GB_2 in the curve $\eta = 1$. The stabilizing-correction method yields the complete approximation of this equation at the first fractional step. The second condition is used to determine ψ in each of the subdomains GB_1 and GB_2 by the method described in Sec. 3 (Stage III). The Thom condition is employed to relate the vortex and the stream function at the lower boundary of the region GB_2. Figure 5 shows calculation results for $b/a = 0.6$, $\omega_0 = 12$, $\mathrm{G} = 70$, $\mathrm{Ca}^{-1} = 0$, and $\mathrm{Ca}^{-1} = 0.5$. Here b is the distance between the vortices, and a is the distance from the common boundary of subdomains GB_1 and GB_2 to the free surface of GB_1 at the initial moment of time. The position of the free surface at the moment of reaching the position is denoted by the solid curve for $\mathrm{Ca}^{-1} = 0$ and by the dashed curve for $\mathrm{Ca}^{-1} = 0.5$.

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